

# Normal projections in Krein spaces

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## Abstract

Given a complex Krein space  $\mathcal{H}$  with fundamental symmetry  $J$ , the aim of this note is to characterize the set of  $J$ -normal projections

$$\mathcal{Q} = \{Q \in L(\mathcal{H}) : Q^2 = Q \text{ and } Q^\# Q = Q Q^\#\}.$$

The ranges of the projections in  $\mathcal{Q}$  are exactly those subspaces of  $\mathcal{H}$  which are pseudo-regular. For a fixed pseudo-regular subspace  $\mathcal{S}$ , there are infinitely many  $J$ -normal projections onto it, unless  $\mathcal{S}$  is regular. Therefore, most of the material herein is devoted to parametrizing the set of  $J$ -normal projections onto a fixed pseudo-regular subspace  $\mathcal{S}$ .

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## 1 Introduction

It is well-known that a (linear, bounded) projection  $Q$ , acting on a Hilbert space  $\mathcal{H}$ , is normal ( $QQ^* = Q^*Q$ ) if and only if it is selfadjoint ( $Q = Q^*$ ). Therefore, there is a one-to-one correspondence between the closed subspaces of  $\mathcal{H}$  and the normal projections acting on  $\mathcal{H}$ .

On the other hand, if  $\mathcal{K}$  is a Krein space with fundamental symmetry  $J$ , it is easy to find  $J$ -normal projections which are not  $J$ -selfadjoint (see Example 1 in Section 3). For a fixed Krein space  $\mathcal{K}$  with fundamental symmetry  $J$ , the purpose of this work is to describe those projections acting on  $\mathcal{K}$  which are  $J$ -normal, i.e. those  $Q = Q^2 \in L(\mathcal{K})$  satisfying

$$QQ^\# = Q^\# Q,$$

where  $Q^\#$  is the  $J$ -adjoint of  $Q$ .

If  $Q$  is  $J$ -normal, observe that  $E = QQ^\#$  is a  $J$ -selfadjoint projection whose range, hereafter denoted by  $R(E)$ , is contained in  $R(Q)$ . Thus,  $R(Q)$  contains a regular subspace of  $\mathcal{K}$ . On the other hand,  $P = Q(I - Q^\#)$  is a projection with  $R(P) = R(Q) \cap R(Q)^{[\perp]} = R(Q)^\circ$ , i.e.  $R(P)$  is the isotropic part of  $R(Q)$ .

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Also, since  $EP = PE = 0$  it follows that  $Q = E + P$  and

$$R(Q) = R(E)[+]R(P) = R(E)[+]R(Q)^\circ,$$

that is,  $R(Q)$  is a *pseudo-regular* subspace of  $\mathcal{K}$ , see [9] for the terminology. Conversely, it will be shown that every pseudo-regular subspace of  $\mathcal{K}$  admits a  $J$ -normal projection onto it. However, it is not hard to prove that a pseudo-regular subspace may admit infinitely many  $J$ -normal projections onto it (see Example 2 in Section 4).

The importance of pseudo-regular subspaces lies in the fact that they enable to generalize some Pontryagin spaces arguments to general Krein spaces, see [9]. They have also been used as a technical tool for the study of spectral functions (and distributions) for particular classes of operators in Krein spaces [10, 11, 13, 14] and to extend the Beurling-Lax theorem for shifts in indefinite metric spaces [4, 5].

Along this work, different characterizations of  $J$ -normal projections will be developed. Furthermore, for a fixed pseudo-regular subspace  $\mathcal{S}$ , we will present a parametrization for the set of  $J$ -normal projections onto  $\mathcal{S}$ .

In the next section we introduce the basic notations and terminology used in the paper. Section 3 is devoted to describe  $J$ -normal projections. In particular, it is shown that every  $J$ -normal projection  $Q$  admits a unique decomposition  $Q = E + P$  where  $E$  is  $J$ -selfadjoint and  $P$  is a  $J$ -normal projection with  $J$ -neutral range. Then, the main consequences of this decomposition are discussed.

In Section 4 it is shown that a (closed) subspace  $\mathcal{S}$  is the range of a  $J$ -normal projection if and only if it is pseudo-regular, i.e. if  $\mathcal{S} + \mathcal{S}^{[\perp]}$  is closed. Then, although there is not a unique  $J$ -normal projection onto an arbitrary pseudo-regular subspace  $\mathcal{S}$ , a formula for a particular  $J$ -normal projection onto  $\mathcal{S}$  is presented (depending only on the fundamental symmetry  $J$  and the orthogonal projections onto  $\mathcal{S}$  and  $\mathcal{S}^\circ$ ).

Section 5 deals with  $J$ -normal projections onto  $J$ -neutral subspaces. It will be shown that there are infinitely many  $J$ -normal projections onto a prescribed  $J$ -neutral subspace (and their nullspaces can be arbitrarily close). Then, for a fixed  $J$ -neutral subspace  $\mathcal{N}$ , a parametrization for the set of  $J$ -normal projections onto  $\mathcal{N}$  is presented.

Finally, the aim of Section 6 is to present an explicit description of the set of  $J$ -normal projections onto a pseudo-regular subspace  $\mathcal{S}$ . First, it is shown that this set can be decomposed in a disjoint union of decks. Then, considering the projections as block-operator matrices according to an appropriate orthogonal decomposition, each deck is parametrized.

## 2 Preliminaries

**Notation and terminology** Along this work  $\mathcal{H}$  denotes a complex (separable) Hilbert space. If  $\mathcal{K}$  is another Hilbert space then  $L(\mathcal{H}, \mathcal{K})$  is the algebra of bounded linear operators from  $\mathcal{H}$  into  $\mathcal{K}$  and  $L(\mathcal{H}) = L(\mathcal{H}, \mathcal{H})$ . The

group of linear invertible operators acting on  $\mathcal{H}$  is denoted by  $GL(\mathcal{H})$ . Also,  $L(\mathcal{H})^+$  denotes the cone of positive semidefinite operators acting on  $\mathcal{H}$  and  $GL(\mathcal{H})^+ = GL(\mathcal{H}) \cap L(\mathcal{H})^+$ .

If  $T \in L(\mathcal{H}, \mathcal{K})$  then  $T^* \in L(\mathcal{K}, \mathcal{H})$  denotes the adjoint operator of  $T$ ,  $R(T)$  stands for its range and  $N(T)$  for its nullspace.

Given two closed subspaces  $\mathcal{S}$  and  $\mathcal{T}$  of a Hilbert space  $\mathcal{H}$ ,  $\mathcal{S} \dot{+} \mathcal{T}$  denotes the direct sum of them. On the other hand,  $\mathcal{S} \oplus \mathcal{T}$  stands for their (direct) orthogonal sum and  $\mathcal{S} \ominus \mathcal{T} := \mathcal{S} \cap (\mathcal{S} \cap \mathcal{T})^\perp$ . If  $\mathcal{H} = \mathcal{S} \dot{+} \mathcal{T}$ , there exists a (unique) bounded projection with range  $\mathcal{S}$  and nullspace  $\mathcal{T}$ . Hereafter, it is denoted by  $P_{\mathcal{S} // \mathcal{T}}$ . If  $P_{\mathcal{S}}$  and  $P_{\mathcal{T}}$  stand for the orthogonal projections onto  $\mathcal{S}$  and  $\mathcal{T}$ , respectively,  $P_{\mathcal{S} // \mathcal{T}}$  can be represented as:

$$P_{\mathcal{S} // \mathcal{T}} = P_{\mathcal{S}}(P_{\mathcal{S}} + P_{\mathcal{T}})^{-1}, \quad (2.1)$$

see [2, Lemma 3.1].

Given two closed subspaces  $\mathcal{S}$  and  $\mathcal{T}$  of a Hilbert space  $\mathcal{H}$ , the cosine of the *Friedrichs angle* between  $\mathcal{S}$  and  $\mathcal{T}$  is defined by

$$c(\mathcal{S}, \mathcal{T}) = \sup\{|\langle x, y \rangle| : x \in \mathcal{S} \ominus \mathcal{T}, \|x\| = 1, y \in \mathcal{T} \ominus \mathcal{S}, \|y\| = 1\}.$$

It is well known that

$$c(\mathcal{S}, \mathcal{T}) < 1 \Leftrightarrow \mathcal{S} + \mathcal{T} \text{ is closed} \Leftrightarrow c(\mathcal{S}^\perp, \mathcal{T}^\perp) < 1.$$

Furthermore, if  $P_{\mathcal{S}}$  and  $P_{\mathcal{T}}$  are the orthogonal projections onto  $\mathcal{S}$  and  $\mathcal{T}$ , respectively, then  $c(\mathcal{S}, \mathcal{T}) < 1$  if and only if  $(I - P_{\mathcal{S}})P_{\mathcal{T}}$  has closed range.

On the other hand, the *Dixmier (or minimal) angle* between  $\mathcal{S}$  and  $\mathcal{T}$  is defined by

$$c_0(\mathcal{S}, \mathcal{T}) = \sup\{|\langle x, y \rangle| : x \in \mathcal{S}, \|x\| = 1, y \in \mathcal{T}, \|y\| = 1\}.$$

It is clear that  $c(\mathcal{S}, \mathcal{T}) \leq c_0(\mathcal{S}, \mathcal{T})$ , and if  $\mathcal{S} \cap \mathcal{T} = \{0\}$  then  $c(\mathcal{S}, \mathcal{T}) = c_0(\mathcal{S}, \mathcal{T})$ .

*Remark 2.1.* If  $P_{\mathcal{S}}$  and  $P_{\mathcal{T}}$  are the orthogonal projections onto  $\mathcal{S}$  and  $\mathcal{T}$ , respectively, then

$$c_0(\mathcal{S}, \mathcal{T}) = \|P_{\mathcal{S}}P_{\mathcal{T}}\|.$$

Also,  $\mathcal{H} = \mathcal{S} \dot{+} \mathcal{T}$  if and only if  $\|P_{\mathcal{S}^\perp}P_{\mathcal{T}^\perp}\| < 1$ . See [8] for further details.

## Krein spaces

In what follows we present the standard notation and some basic results on Krein spaces. For a complete exposition on the subject see [6, 12, 1].

Given a Krein space  $(\mathcal{H}, [\cdot, \cdot])$  with a *fundamental decomposition*  $\mathcal{H} = \mathcal{H}_+ \dot{+} \mathcal{H}_-$ , the direct (orthogonal) sum of the Hilbert spaces  $(\mathcal{H}_+, [\cdot, \cdot])$  and  $(\mathcal{H}_-, -[\cdot, \cdot])$  is denoted by  $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ .

Observe that the indefinite metric and the inner product of  $\mathcal{H}$  are related by means of a *fundamental symmetry*, i.e. a unitary selfadjoint operator  $J \in L(\mathcal{H})$  which satisfies:

$$[x, y] = \langle Jx, y \rangle, \quad x, y \in \mathcal{H}.$$

If  $\mathcal{H}$  and  $\mathcal{K}$  are Krein spaces,  $L(\mathcal{H}, \mathcal{K})$  stands for the vector space of linear transformations which are bounded with respect to the associated Hilbert spaces  $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$  and  $(\mathcal{K}, \langle \cdot, \cdot \rangle_{\mathcal{K}})$ . Given  $T \in L(\mathcal{H}, \mathcal{K})$ , the  $J$ -adjoint operator of  $T$  is defined by  $T^{\#} = J_{\mathcal{H}} T^* J_{\mathcal{K}}$ , where  $J_{\mathcal{H}}$  and  $J_{\mathcal{K}}$  are the fundamental symmetries associated to  $\mathcal{H}$  and  $\mathcal{K}$ , respectively. An operator  $T \in L(\mathcal{H})$  is  $J$ -selfadjoint if  $T = T^{\#}$ .

A vector  $x \in \mathcal{H}$  is  $J$ -positive if  $[x, x] > 0$ . A subspace  $\mathcal{S}$  of  $\mathcal{H}$  is  $J$ -positive if every  $x \in \mathcal{S}$ ,  $x \neq 0$ , is a  $J$ -positive vector.  $J$ -nonnegative,  $J$ -neutral,  $J$ -negative and  $J$ -nonpositive vectors and subspaces are defined analogously.

Given a subspace  $\mathcal{S}$  of a Krein space  $\mathcal{H}$ , the  $J$ -orthogonal complement to  $\mathcal{S}$  is defined by

$$\mathcal{S}^{[\perp]} = \{x \in \mathcal{H} : [x, s] = 0, \text{ for every } s \in \mathcal{S}\}.$$

Usually,  $\mathcal{S}^{\circ} := \mathcal{S} \cap \mathcal{S}^{[\perp]}$  (the *isotropic part of  $\mathcal{S}$* ) is a non-trivial subspace. Then, a subspace  $\mathcal{S}$  of  $\mathcal{H}$  is  $J$ -non-degenerated if  $\mathcal{S} \cap \mathcal{S}^{[\perp]} = \{0\}$ . Otherwise, it is a  $J$ -degenerated subspace of  $\mathcal{H}$ .

**Definition.** A subspace  $\mathcal{S}$  of a Krein space  $\mathcal{H}$  is a *regular subspace* if it is the range of a  $J$ -selfadjoint projection, i.e. if there exists  $E \in L(\mathcal{H})$  such that  $E = E^2 = E^{\#}$  and  $R(E) = \mathcal{S}$ .

Given a regular subspace  $\mathcal{S}$ , observe that  $\mathcal{S}^{[\perp]}$  is the nullspace of the  $J$ -selfadjoint projection  $E$  onto  $\mathcal{S}$ . Furthermore, if  $P$  is the orthogonal projection onto  $\mathcal{S}$ , the orthogonal projection onto  $\mathcal{S}^{[\perp]}$  coincides with  $J(I - P)J$ . Thus, by (2.1), it follows that

$$E = P(P + I - JPJ)^{-1}, \quad (2.2)$$

see [3] for another formula for  $E$ .

**Proposition 2.2** ([3]). *A closed subspace  $\mathcal{S}$  is regular if and only if*

$$\|PJ(I - P)\| < 1,$$

*or equivalently  $(I - P)JPJ(I - P) \leq (1 - \varepsilon)I$  for some  $\varepsilon > 0$ , where  $P$  is the orthogonal projection onto  $\mathcal{S}$ .*

The following result seems to be well known, however its proof is included for the sake of completeness.

**Lemma 2.3.** *Let  $Q \in L(\mathcal{H})$  be a projection acting on a Krein space  $\mathcal{H}$  with fundamental symmetry  $J$ . Then, the following conditions are equivalent:*

1.  $Q^{\#}Q = 0$ ;
2.  $R(Q)$  is a  $J$ -neutral subspace;
3.  $PJP = 0$ , where  $P$  is the orthogonal projection onto  $R(Q)$ ;

4. the orthogonal projection  $P$  onto  $R(Q)$  admits the representation (according to the fundamental decomposition  $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ )

$$P = \frac{1}{2} \begin{pmatrix} V^*V & V^* \\ V & VV^* \end{pmatrix},$$

where  $V \in L(\mathcal{H}_+, \mathcal{H}_-)$  is a partial isometry.

*Proof.* The equivalences 1.  $\leftrightarrow$  2.  $\leftrightarrow$  3.  $\leftrightarrow$  4. and the implication 5.  $\rightarrow$  1. are easy to check. On the other hand, if  $\mathcal{S} = R(Q)$  is a  $J$ -neutral subspace of  $\mathcal{H}$  then its angular operator  $V \in L(\mathcal{H}_+, \mathcal{H}_-)$  is a partial isometry. Therefore

$$\begin{aligned} \mathcal{S} &= \{(x_+, Vx_+) \in \mathcal{H}_+ \oplus \mathcal{H}_- : x_+ \in P_+(\mathcal{S}) = N(V)^\perp\} \\ &= \{(V^*Vu, Vu) \in \mathcal{H}_+ \oplus \mathcal{H}_- : u \in \mathcal{H}_+\} = R\left(\begin{bmatrix} V & V^* \\ V & V \end{bmatrix}\right), \end{aligned}$$

see [12, Ch. 1, §8]. Then, since  $V$  is a partial isometry, the operator

$$P = \frac{1}{2} \begin{pmatrix} V^*V & V^* \\ V & VV^* \end{pmatrix},$$

satisfies  $P^2 = P = P^*$ , i.e.  $P$  is the orthogonal projection onto  $\mathcal{S}$ .  $\square$

### 3 Decompositions of a $J$ -normal projection

Every normal projection acting on a Hilbert space is selfadjoint. However, the following example shows that there are  $J$ -normal projections acting on a Krein space (i.e. projections that commute with its  $J$ -adjoint) which are not  $J$ -selfadjoint.

*Example 1.* If  $\mathbb{C}^3$  is endowed with the indefinite inner product  $[x, y] = x_1\overline{y_1} + x_2\overline{y_2} - x_3\overline{y_3}$ , where  $x = (x_1, x_2, x_3)$ ,  $y = (y_1, y_2, y_3) \in \mathbb{C}^3$ , consider the projection  $Q$  whose matrix representation in the canonical basis is given by

$$Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Then, it is easy to see that

$$Q^\# = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \neq Q \quad \text{and} \quad QQ^\# = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = Q^\#Q.$$

In what follows, the basic properties of  $J$ -normal projections are developed.

**Theorem 3.1.** *Given a projection  $Q \in L(\mathcal{H})$ ,  $Q$  is  $J$ -normal if and only if there exist a  $J$ -selfadjoint projection  $E \in L(\mathcal{H})$  and a projection  $P \in L(\mathcal{H})$  satisfying  $PP^\# = P^\#P = 0$  such that*

$$Q = E + P. \tag{3.1}$$

*The projections  $E$  and  $P$  are uniquely determined by  $Q$ .*

*Proof.* If  $Q \in L(\mathcal{H})$  is a  $J$ -normal projection, then  $E = QQ^\#$  is a  $J$ -selfadjoint projection. Notice that  $P := Q(I - Q^\#)$  is also a projection and, since  $I - Q$  is also  $J$ -normal, it holds that

$$PP^\# = Q(I - Q^\#)(I - Q)Q^\# = Q(I - Q)(I - Q^\#)Q^\# = 0.$$

In the same way,  $P^\#P = 0$ .

Conversely, suppose that  $Q = E + P$  where  $E$  is  $J$ -selfadjoint and  $P$  is a projection satisfying  $PP^\# = P^\#P = 0$ . Since  $Q^2 = Q$ , it follows that  $EP + PE = 0$ . Notice that  $R(E) \cap R(P) = \{0\}$ . In fact, if  $x \in R(E) \cap R(P)$  it is easy to see that  $0 = (EP + PE)x = 2x$ . So,  $x = 0$ . Therefore,  $EP = PE = 0$  (and  $EP^\# = P^\#E = 0$ ).

Thus, recalling that  $PP^\# = P^\#P = 0$  it follows easily that  $QQ^\# = Q^\#Q = E$ , i.e.  $Q$  is  $J$ -normal. Notice that  $P = Q - E = Q(I - Q^\#)$ . The uniqueness of this decomposition follows from the last part of the proof.  $\square$

If  $Q \in L(\mathcal{H})$  is a  $J$ -normal projection, notice that the (uniquely) determined projections in the decomposition of Theorem 3.1 are

$$E = QQ^\# \quad \text{and} \quad P = Q(I - Q^\#). \quad (3.2)$$

Throughout this paper,  $E$  and  $P$  will be referred as the *regular part* and the *neutral part* of  $Q$ , respectively.

**Corollary 3.2.** *Let  $Q \in L(\mathcal{H})$  be a  $J$ -normal projection. Then,  $Q$  is  $J$ -selfadjoint if and only if  $R(Q)^\circ$  is trivial.*

*Proof.* Observe that  $Q$  is  $J$ -selfadjoint if and only if  $Q = QQ^\#$ , or equivalently,  $P = Q(I - Q^\#) = 0$ . But  $R(P) = R(Q) \cap N(Q^\#) = R(Q)^\circ$ . So,  $P = 0$  if and only if  $R(Q)^\circ = \{0\}$ .  $\square$

**Corollary 3.3.** *Given a projection  $Q \in L(\mathcal{H})$ ,  $Q$  is  $J$ -normal if and only if*

$$Q = GH,$$

*where  $G \in L(\mathcal{H})$  is a  $J$ -selfadjoint projection and  $H \in L(\mathcal{H})$  is a  $J$ -normal projection with  $J$ -neutral kernel contained in  $R(G)$ . Furthermore, this factorization is unique and the projections  $G$  and  $H$  commute.*

*Proof.* If  $Q$  is  $J$ -normal, then  $G = I - (I - Q)(I - Q)^\#$  and  $H = I - (I - Q)Q^\#$  satisfy the desired properties.

Conversely, if  $Q = GH$  for a pair of projections  $G$  and  $H$  satisfying the assumptions, notice that  $(I - G)(I - H) = 0$ , or equivalently,  $I + GH = G + H$ . Thus,

$$I - Q = I - GH = (I - G) + (I - H),$$

$I - G$  is  $J$ -selfadjoint and  $I - H$  satisfies  $(I - H)(I - H)^\# = (I - H)^\#(I - H) = 0$ . Then, by Theorem 3.1,  $Q$  is  $J$ -normal.

The uniqueness of the factorization and the commutativity of  $G$  and  $H$  also follow from the above theorem.  $\square$

**Corollary 3.4.** *If  $Q \in L(\mathcal{H})$  is a  $J$ -normal projection and  $Q = E + P$  is the decomposition given by Theorem 3.1, then there exists a unique  $J$ -selfadjoint projection  $F \in L(\mathcal{H})$  such that*

$$I - Q = F + P^\#. \quad (3.3)$$

Moreover,  $EF = 0$ .

*Proof.* Applying Theorem 3.1 to  $I - Q$  it follows that its  $J$ -selfadjoint part is  $F = (I - Q)(I - Q)^\#$  and

$$(I - Q) - F = (I - Q) - (I - Q)(I - Q)^\# = (I - Q)Q^\# = P^\#.$$

Furthermore,  $E = QQ^\# = Q^\#Q$  and then it is obvious that  $EF = 0$ .  $\square$

**Lemma 3.5.** *Let  $Q \in L(\mathcal{H})$  be a  $J$ -normal projection and consider the neutral part  $P \in L(\mathcal{H})$  of  $Q$ . Then,*

$$R(P) = R(Q)^\circ \quad \text{and} \quad R(P^\#) = N(Q)^\circ. \quad (3.4)$$

Therefore,  $R(Q)^\circ$  and  $N(Q)^\circ$  have the same dimension and codimension.

*Proof.* Indeed, if  $Q$  is  $J$ -normal then  $P = Q(I - Q^\#) = (I - Q^\#)Q$  and

$$R(P) = R(Q) \cap N(Q^\#) = R(Q) \cap R(Q)^{[\perp]} = R(Q)^\circ.$$

The assertion on  $R(P^\#)$  follows analogously. Finally, notice that

$$\begin{aligned} \dim R(Q)^\circ &= \dim R(P) = \dim N(P)^\perp = \dim R(P^*) = \dim R(P^\#) \\ &= \dim N(Q)^\circ, \end{aligned}$$

and  $\text{codim } R(Q)^\circ = \dim N(P) = \dim R(P)^\perp = \dim N(P^*) = \dim N(P^\#) = \text{codim } N(Q)^\circ$ .  $\square$

*Remark 3.6.* Let  $Q \in L(\mathcal{H})$  be a  $J$ -normal projection with decompositions  $Q = E + P$  and  $I - Q = F + P^\#$ . From the  $J$ -normality of  $Q$  and the formulas

$$E = QQ^\#, \quad P = Q(I - Q^\#), \quad F = (I - Q)(I - Q)^\# \quad \text{and} \quad PE = PF = 0,$$

the following facts are easily deduced:

1.  $R(E) = R(Q) \cap R(Q^\#)$  and  $R(F) = N(Q) \cap N(Q^\#)$ . Moreover,

$$R(Q) = R(E) \dot{+} R(P) \quad \text{and} \quad N(Q) = R(F) \dot{+} R(P^\#).$$

2. Also, since  $PP^\# = P^\#P = 0$ , observe that  $P + P^\#$  is a  $J$ -selfadjoint projection with range  $R(Q)^\circ \dot{+} N(Q)^\circ$ . Therefore,  $R(Q)^\circ \dot{+} N(Q)^\circ$  is regular.

3. Finally, by the items above, notice that

$$\mathcal{H} = R(Q) \dot{+} N(Q) = (R(E)[\dot{+}] R(P)) \dot{+} (R(F)[\dot{+}] R(P^\#)).$$

Then, if  $Q$  is  $J$ -normal,  $\mathcal{H}$  can be decomposed as

$$\mathcal{H} = R(Q) \cap R(Q^\#) [\dot{+}] (R(Q)^\circ \dot{+} N(Q)^\circ) [\dot{+}] N(Q) \cap N(Q^\#). \quad (3.5)$$

In fact, (3.5) is equivalent to the  $J$ -normality of  $Q$ .

**Proposition 3.7.** *Let  $Q \in L(\mathcal{H})$  be a projection. Then,  $Q$  is  $J$ -normal if and only if*

$$\mathcal{H} = R(Q) \cap R(Q^\#) \dot{+} R(Q) \cap N(Q^\#) \dot{+} N(Q) \cap R(Q^\#) \dot{+} N(Q) \cap N(Q^\#). \quad (3.6)$$

*Proof.* If  $Q$  is  $J$ -normal, the decomposition follows from item 3. in the above remark. Conversely, suppose that (3.6) holds. Given  $x \in \mathcal{H}$  there exist (unique)  $x_1 \in R(Q) \cap R(Q^\#)$ ,  $x_2 \in R(Q) \cap N(Q^\#)$ ,  $x_3 \in N(Q) \cap R(Q^\#)$  and  $x_4 \in N(Q) \cap N(Q^\#)$  such that  $x = x_1 + x_2 + x_3 + x_4$ . Then,

$$Q^\# Qx = Q^\#(x_1 + x_2) = x_1 = Q(x_1 + x_3) = QQ^\#x.$$

Therefore,  $Q^\# Qx = QQ^\#x$  for every  $x \in \mathcal{H}$ , i.e.  $Q$  is  $J$ -normal.  $\square$

## 4 The range of a $J$ -normal projection

The aim of this section is to characterize the ranges of the family of  $J$ -normal projections acting on a Krein space. The main result in this direction addresses the fact that a (closed) subspace is the range of a  $J$ -normal projection if and only if it is a pseudo-regular subspace. Thus, the first paragraphs are devoted to recall the definition of pseudo-regularity and to state some well known equivalent conditions. Throughout this section,  $\mathcal{H}$  denotes a Krein space with fundamental symmetry  $J$ .

**Definition.** A closed subspace  $\mathcal{S}$  of  $\mathcal{H}$  is called *pseudo-regular* if the algebraic sum  $\mathcal{S} + \mathcal{S}^{[\perp]}$  is closed.

The following proposition compiles several conditions which are equivalent to pseudo-regularity. These facts are well known but they are scattered throughout the literature and different research papers, e.g. see [12, 5, 9, 13].

**Proposition 4.1.** *Let  $\mathcal{S}$  be a closed subspace of  $\mathcal{H}$  and consider its Gramian operator  $G_{\mathcal{S}} = P_{\mathcal{S}}J|_{\mathcal{S}} : \mathcal{S} \rightarrow \mathcal{S}$ . Then, the following conditions are equivalent:*

1.  $\mathcal{S}$  is pseudo-regular.
2.  $(\mathcal{S}^\circ)^{[\perp]} = \mathcal{S} + \mathcal{S}^{[\perp]}$ .
3. There exists a regular subspace  $\mathcal{M}$  such that  $\mathcal{S} = \mathcal{S}^\circ[\dot{+}] \mathcal{M}$ .



4. If  $\mathcal{S} = \mathcal{T} \dot{+} \mathcal{S}^\circ$ , then  $\mathcal{T}$  is regular.
5. There exists a regular subspace  $\mathcal{N} \supseteq \mathcal{S}$  such that  $\mathcal{S}^\circ = \mathcal{N} \cap \mathcal{S}^{[\perp]}$ .
6.  $\mathcal{S}/\mathcal{S}^\circ$  is a Krein space.
7. 0 is an isolated point of  $\sigma(G_{\mathcal{S}})$ .

**Proposition 4.2** (T. Ando). *Given a (closed) subspace  $\mathcal{S}$  of  $\mathcal{H}$ , consider its isotropic part  $\mathcal{S}^\circ$ . Let  $P$  and  $P_0$  denote the orthogonal projections onto  $\mathcal{S}$  and  $\mathcal{S}^\circ$ , respectively. Then,  $\mathcal{S}$  is pseudo-regular if and only if*

$$\|(P - P_0)J(I - P)\| < 1.$$

*Proof.* Observe that  $J(I - P)J$  is the orthogonal projection onto  $\mathcal{S}^{[\perp]}$ . By definition,  $\mathcal{S}$  is pseudo-regular if

$$\mathcal{S} + \mathcal{S}^{[\perp]} \text{ is closed.}$$

But  $\mathcal{S} + \mathcal{S}^{[\perp]}$  is closed if and only if  $c(\mathcal{S}, \mathcal{S}^{[\perp]}) < 1$ . Also, notice that  $c(\mathcal{S}, \mathcal{S}^{[\perp]}) = c_0(\mathcal{S} \ominus \mathcal{S}^\circ, \mathcal{S}^{[\perp]}) = \|(P - P_0)J(I - P)J\|$  (see the Preliminaries). Hence,  $\mathcal{S}$  is pseudo-regular if and only if

$$\|(P - P_0)J(I - P)\| < 1. \quad \square$$

**Theorem 4.3.** *Let  $\mathcal{S}$  be a closed subspace of  $\mathcal{H}$ . Then,  $\mathcal{S}$  is the range of a  $J$ -normal projection if and only if  $\mathcal{S}$  is a pseudo-regular subspace of  $\mathcal{H}$ .*

*Proof.* If  $\mathcal{S}$  is the range of a  $J$ -normal projection  $Q$  then, by Remark 3.6,  $\mathcal{S} = R(E)[\dot{+}] \mathcal{S}^\circ$  where  $E = QQ^\#$ . Furthermore,  $R(E)$  is regular because  $E$  is a  $J$ -selfadjoint projection. Thus,  $\mathcal{S}$  is a pseudo-regular subspace.

Conversely, suppose that  $\mathcal{S}$  is a pseudo-regular subspace and let  $P$  be the orthogonal projection onto the isotropic subspace  $\mathcal{S}^\circ$ . Since  $R(P)$  is  $J$ -neutral, it follows by Lemma 2.3 that  $PJP = 0$ . Then,  $PP^\# = P^\#P = 0$ .

Consider the subspace  $\mathcal{T} = \mathcal{S} \ominus \mathcal{S}^\circ$ . Since  $\mathcal{S} = \mathcal{T}[\dot{+}] \mathcal{S}^\circ$ , Proposition 4.1 assures that  $\mathcal{T}$  is a regular subspace of  $\mathcal{H}$ . Thus, there is a (unique)  $J$ -selfadjoint projection  $E$  with  $R(E) = \mathcal{T}$ .

Furthermore,  $PE = EP = 0$  because  $\mathcal{T} \subset (\mathcal{S}^\circ)^\perp$  and  $\mathcal{S}^\circ \subset \mathcal{S}^{[\perp]} \subset \mathcal{T}^{[\perp]}$ . Then  $Q = E + P$  is also a projection with

$$R(Q) = R(E) + R(P) = \mathcal{T} \dot{+} \mathcal{S}^\circ = \mathcal{S}.$$

Finally, the  $J$ -normality of  $Q$  follows from Theorem 3.1.  $\square$

Recall that if  $\kappa = \min\{\dim \mathcal{H}_+, \dim \mathcal{H}_-\} < \infty$ , the Krein space with fundamental decomposition  $\mathcal{H} = \mathcal{H}_+ \dot{+} \mathcal{H}_-$  is called a *Pontryagin space* and is denoted by  $\Pi_\kappa$ . In a Pontryagin space  $\Pi_\kappa$ , a closed subspace  $\mathcal{S}$  is regular if and only if it is  $J$ -non-degenerated (see e.g. [12]). Thus, every  $J$ -non-degenerated subspace of  $\Pi_\kappa$  admits a (unique)  $J$ -selfadjoint projection onto it. Furthermore,

**Corollary 4.4.** *If  $\Pi_\kappa$  is a Pontryagin space, then every closed subspace  $\mathcal{S}$  of  $\Pi_\kappa$  admits a  $J$ -normal projection onto it.*

*Proof.* Since  $\mathcal{S}^\circ$  is a closed subspace of  $\mathcal{S}$ ,  $\mathcal{S}$  can be written as

$$\mathcal{S} = \mathcal{S}^\circ \oplus (\mathcal{S} \ominus \mathcal{S}^\circ).$$

Furthermore,  $\mathcal{T} := \mathcal{S} \ominus \mathcal{S}^\circ$  is  $J$ -orthogonal to  $\mathcal{S}^\circ$ . Hence,  $\mathcal{S} = \mathcal{S}^\circ [+] \mathcal{T}$ . It is easy to see that  $\mathcal{T}$  is a  $J$ -non-degenerated subspace of  $\mathcal{H}$  and therefore,  $\mathcal{T}$  is regular because  $\Pi_\kappa$  is a Pontryagin space. Thus,  $\mathcal{S}$  is the direct sum of its isotropic part and a regular subspace and, by Theorem 4.3,  $\mathcal{S}$  is the range of a  $J$ -normal projection.  $\square$

The last paragraphs of this section are devoted to discussing the non-uniqueness of  $J$ -normal projections associated to a pseudo-regular subspace. First of all, observe the following example.

*Example 2.* As in Example 1, consider the Minkowski space  $(\mathbb{C}^3, [ , ])$ . Fix  $\mathcal{S}$  by  $\mathcal{S} = \text{span}\{(1, 0, 0), (0, 1, 1)\}$ . Given a vector  $v = (x, y, z) \in \mathbb{C}^3 \setminus \mathcal{S}$ , let  $Q_v$  be the projection onto  $\mathcal{S}$  along the subspace spanned by  $v$ . According to the canonical basis of  $\mathbb{C}^3$ , its matrix representation is

$$Q_v = \frac{1}{z - y} \begin{pmatrix} z - y & x & -x \\ 0 & z & -y \\ 0 & z & -y \end{pmatrix}.$$

A few calculations show that

$$Q_v^\# = \frac{1}{\overline{z - y}} \begin{pmatrix} \overline{z - y} & 0 & 0 \\ \overline{x} & \overline{z} & -\overline{z} \\ \overline{x} & \overline{y} & -\overline{y} \end{pmatrix}.$$

Then, it is easy to see that

$$\begin{aligned} Q_v^\# Q_v &= \frac{1}{|z - y|^2} \begin{pmatrix} |z - y|^2 & x \overline{(z - y)} & -x \overline{(z - y)} \\ \overline{x}(z - y) & |x|^2 & -|x|^2 \\ \overline{x}(z - y) & |x|^2 & -|x|^2 \end{pmatrix} \quad \text{and} \\ Q_v Q_v^\# &= \frac{1}{|z - y|^2} \begin{pmatrix} |z - y|^2 & x \overline{(z - y)} & -x \overline{(z - y)} \\ \overline{x}(z - y) & |z|^2 - |y|^2 & -|z|^2 + |y|^2 \\ \overline{x}(z - y) & |z|^2 - |y|^2 & -|z|^2 + |y|^2 \end{pmatrix}. \end{aligned}$$

Therefore,  $Q_v$  is a  $J$ -normal projection onto  $\mathcal{S}$  if and only if  $|z|^2 = |x|^2 + |y|^2$ .

The above example also shows that, for a fixed projection  $Q \in L(\mathcal{H})$ , the idempotency of the  $J$ -selfadjoint operators  $QQ^\#$  and  $Q^\#Q$  is not a sufficient condition for the  $J$ -normality of  $Q$ . In fact, notice that  $Q_v^\# Q_v$  and  $Q_v Q_v^\#$  are projections for every  $v \in \mathbb{C}^3 \setminus \mathcal{S}$ , even if  $|z|^2 \neq |x|^2 + |y|^2$ .

Although there is not a unique  $J$ -normal projection onto a fixed arbitrary pseudo-regular subspace  $\mathcal{S}$ , it is possible to present a particular  $J$ -normal projection onto  $\mathcal{S}$  in terms of the orthogonal projections onto  $\mathcal{S}$  and  $\mathcal{S}^\circ$ . Observe that this particular  $J$ -normal projection onto  $\mathcal{S}$  is the one discussed in Theorem 4.3.

**Corollary 4.5.** *Given a (closed) pseudo-regular subspace  $\mathcal{S}$  of  $\mathcal{H}$ , let  $P$  and  $P_0$  denote the orthogonal projections onto  $\mathcal{S}$  and  $\mathcal{S}^\circ$ , respectively. Then,*

$$Q = (P - P_0)(P - P_0 + I - J(P - P_0)J)^{-1} + P_0, \quad (4.1)$$

*is a  $J$ -normal projection onto  $\mathcal{S}$ .*

*Proof.* Since  $\mathcal{S} \ominus \mathcal{S}^\circ$  is a regular subspace of  $\mathcal{H}$ , the  $J$ -selfadjoint projection  $E$  onto  $\mathcal{S} \ominus \mathcal{S}^\circ$  can be written as

$$E = (P - P_0)(P - P_0 + I - J(P - P_0)J)^{-1},$$

see (2.2). Furthermore, by Theorem 3.1,  $Q = E + P_0 = (P - P_0)(P - P_0 + I - J(P - P_0)J)^{-1} + P_0$  is a  $J$ -normal projection onto  $\mathcal{S}$ .  $\square$

## 5 $J$ -normal projections with $J$ -neutral range

From now on, every subspace considered is assumed to be closed.

As it was shown in the previous section, a pseudo-regular subspace may admit infinitely many  $J$ -normal projections onto it. In order to provide a parametrization of the set of  $J$ -normal projections onto a prescribed pseudo-regular subspace, consider the simplest case first, i.e. a  $J$ -neutral subspace. This section is devoted to studying  $J$ -normal projections onto  $J$ -neutral subspaces, i.e. those projections  $P \in L(\mathcal{H})$  satisfying  $PP^\# = P^\#P = 0$ .

It is obvious that every  $J$ -neutral subspace  $\mathcal{N}$  of a Krein space  $\mathcal{H}$  is a pseudo-regular one, since  $\mathcal{N} = \mathcal{N}^\circ$ . In particular,

**Lemma 5.1.** *If  $\mathcal{N}$  is a  $J$ -neutral subspace then the orthogonal projection  $P := P_{\mathcal{N}} \in L(\mathcal{H})$  is  $J$ -normal. Furthermore,  $PP^\# = P^\#P = 0$ .*

*Proof.* By Lemma 2.3, the assumption on  $\mathcal{N}$  is equivalent to  $PJP = 0$ . Thus,

$$PP^\# = PJP \cdot J = 0 \quad \text{and} \quad P^\#P = J \cdot PJP = 0. \quad \square$$

**Proposition 5.2.** *Let  $\mathcal{N}_1$  and  $\mathcal{N}_2$  be (closed)  $J$ -neutral subspaces of  $\mathcal{H}$  such that  $\mathcal{N}_1 \cap \mathcal{N}_2 = \{0\}$ . Then, the following conditions are equivalent:*

1. *there exists a  $J$ -normal projection  $P \in L(\mathcal{H})$  such that  $R(P) = \mathcal{N}_1$  and  $R(P^\#) = \mathcal{N}_2$ ;*
2.  *$\mathcal{N}_1 + \mathcal{N}_2$  is regular;*
3.  *$\mathcal{N}_1 \dot{+} \mathcal{N}_2^{[\perp]} = \mathcal{H}$ .*

*Proof.* 1.  $\Rightarrow$  2. follows from item 2. of Remark 3.6.

2.  $\Rightarrow$  3.: Suppose that  $\mathcal{M} = \mathcal{N}_1 + \mathcal{N}_2$  is regular. Then,  $\mathcal{M}^{[\perp]} = \mathcal{N}_1^{[\perp]} \cap \mathcal{N}_2^{[\perp]}$  is also regular and

$$\mathcal{H} = \mathcal{M} \dot{+} \mathcal{M}^{[\perp]} = \mathcal{N}_1 \dot{+} (\mathcal{N}_2 \dot{+} \mathcal{N}_1^{[\perp]} \cap \mathcal{N}_2^{[\perp]}) \subseteq \mathcal{N}_1 + \mathcal{N}_2^{[\perp]},$$

because  $\mathcal{N}_2$  is  $J$ -neutral. Analogously,  $\mathcal{H} = \mathcal{N}_1^{[\perp]} + \mathcal{N}_2$  and  $\mathcal{N}_1 \cap \mathcal{N}_2^{[\perp]} = (\mathcal{N}_1^{[\perp]} + \mathcal{N}_2)^{[\perp]} = \{0\}$ . Thus,  $\mathcal{H} = \mathcal{N}_1 \dot{+} \mathcal{N}_2^{[\perp]}$ .

3.  $\Rightarrow$  1.: If  $\mathcal{N}_1 \dot{+} \mathcal{N}_2^{[\perp]} = \mathcal{H}$ , consider the projection  $P := P_{\mathcal{N}_1 // \mathcal{N}_2^{[\perp]}}$ . Then,  $P^\# = P_{\mathcal{N}_2 // \mathcal{N}_1^{[\perp]}}$  and it is easy to see that  $PP^\# = P^\#P = 0$ . Therefore,  $P$  is a  $J$ -normal projection with  $R(P) = \mathcal{N}_1$  and  $R(P^\#) = \mathcal{N}_2$ .  $\square$

As a consequence of the above proposition, if  $P$  is a  $J$ -normal projection onto a  $J$ -neutral subspace, the subspaces  $R(P)$  and  $R(P^\#)$  are *skewly linked* (see [12, Def. 1.29]). Moreover, in a Pontryagin space  $\Pi_\kappa$ , a pair of  $J$ -neutral subspaces  $\mathcal{N}_1, \mathcal{N}_2$  of  $\Pi_\kappa$  is skewly linked if and only if there exists a  $J$ -normal projection  $P \in L(\mathcal{H})$  such that  $R(P) = \mathcal{N}_1$  and  $R(P^\#) = \mathcal{N}_2$ .

*Remark 5.3.* If  $\mathcal{N}$  is a  $J$ -neutral subspace then  $\mathcal{N} + J(\mathcal{N})$  is regular. In fact, by Lemma 5.1, the orthogonal projection  $P$  onto  $\mathcal{N}$  is a  $J$ -normal projection and  $R(P^\#) = J(\mathcal{N})$ . So, by the above proposition,  $\mathcal{N} + J(\mathcal{N})$  is regular.

**Proposition 5.4.** *Let  $Q \in L(\mathcal{H})$  be a projection such that  $R(Q)^\circ + N(Q)^\circ$  is regular. Then, there exist projections  $E, P \in L(\mathcal{H})$  such that  $PP^\# = P^\#P = 0$  and*

$$Q = E + P.$$

*Proof.* By Proposition 5.2,  $\mathcal{H}$  can be decomposed as  $\mathcal{H} = R(Q)^\circ + (N(Q)^\circ)^{[\perp]}$  and  $P = P_{R(Q)^\circ // (N(Q)^\circ)^{[\perp]}}$  is  $J$ -normal. Since  $R(P) \subseteq R(Q)$ , it follows that  $QP = P$ . Also,  $PQ$  is a projection and  $R(PQ) = R(P)$ . Furthermore,

$$\begin{aligned} N(PQ) &= N(Q) + R(Q) \cap N(P) = N(Q) + R(Q) \cap (N(Q)^\circ)^{[\perp]} \\ &\subseteq (N(Q)^\circ)^{[\perp]} = N(P). \end{aligned}$$

Thus,  $PQ = P$  and  $E := Q - P$  is a projection because of

$$E^2 = Q - QP - PQ + P = Q - P - P + P = Q - P = E.$$

Notice that  $PE = EP = 0$  and therefore  $Q = E + P$ .  $\square$

Following the notation of the above proof, observe that  $E = Q - P = Q(I - P) = (I - P)Q$ . Hence,  $R(E) = R(Q) \cap N(P) = R(Q) \cap (N(Q)^\circ)^{[\perp]}$  and  $N(E) = R(P) + N(Q) = R(Q)^\circ + N(Q)$ . Therefore,

$$E = P_{R(Q) \cap (N(Q)^\circ)^{[\perp]} // R(Q)^\circ + N(Q)}.$$

Thus, the following is a sufficient condition to guarantee that the decomposition of the above proposition is the same as in Theorem 3.1.

**Corollary 5.5.** *Let  $Q \in L(\mathcal{H})$  be a projection such that  $R(Q)^\circ + N(Q)^\circ$  is regular. Then, the following conditions are equivalent:*

1.  $Q$  is  $J$ -normal;
2.  $R(Q) \cap (N(Q)^\circ)^{[\perp]} \subseteq R(Q) \cap R(Q^\#)$ ;

$$3. N(Q) \cap (R(Q)^\circ)^{[\perp]} \subseteq N(Q) \cap N(Q^\#).$$

*Proof.* If  $Q$  is  $J$ -normal, then  $N(Q)$  is a pseudo-regular subspace. So,

$$(N(Q)^\circ)^{[\perp]} = N(Q) + N(Q)^{[\perp]} = N(Q) + R(Q^\#).$$

Then, if  $x \in R(Q) \cap (N(Q)^\circ)^{[\perp]}$ , there exist  $u \in N(Q)$  and  $v \in \mathcal{H}$  such that  $x = u + Q^\#v$ . Hence,

$$x = Qx = Q(u + Q^\#v) = QQ^\#v,$$

i.e.  $x \in R(Q) \cap R(Q^\#)$ . Thus,  $R(Q) \cap (N(Q)^\circ)^{[\perp]} \subseteq R(Q) \cap R(Q^\#)$ .

Conversely, suppose that  $R(Q) \cap (N(Q)^\circ)^{[\perp]} \subseteq R(Q) \cap R(Q^\#)$ . Then, consider the decomposition  $Q = E + P$  given by Proposition 5.4, where  $E, P \in L(\mathcal{H})$  are projections and  $PP^\# = P^\#P = 0$ . Observe that

$$R(E) = R(Q) \cap (N(Q)^\circ)^{[\perp]} = R(Q) \cap R(Q^\#),$$

because  $N(Q)^\circ \subseteq N(Q) = R(Q^\#)^{[\perp]}$ . Also,

$$R(E^\#) = N(E)^{[\perp]} = N(Q)^{[\perp]} \cap (R(Q)^\circ)^{[\perp]} \supseteq R(Q^\#) \cap R(Q) = R(E).$$

Thus,  $E^\#E = E$  and, by Theorem 3.1,  $Q$  is  $J$ -normal.

Finally, notice that the equivalence  $1. \leftrightarrow 3.$  follows considering  $I - Q$  instead of  $Q$ .  $\square$

The following result shows that, for a fixed  $J$ -neutral subspace, there are infinitely many  $J$ -normal projections onto it. Furthermore, the nullspaces of these projections can be arbitrarily close.

**Proposition 5.6** (T. Ando). *Suppose that a (non-trivial) projection  $P \in L(\mathcal{H})$  satisfies  $PP^\# = P^\#P = 0$ . Then, there exists a one-parameter family of (different)  $J$ -normal projections  $P_\varepsilon \in L(\mathcal{H})$  onto  $R(P)$  (for  $0 < \varepsilon < \varepsilon_0$ ) such that*

$$\|P_\varepsilon - P\| \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.$$

*Proof.* Let  $P_R$  (resp.  $P_N$ ) be the orthogonal projection onto  $R(P)$  (resp.  $N(P)$ ). Then, the ranges of these projections are  $J$ -neutral subspaces and, by Lemma 2.3, there is a partial isometry  $V \in L(\mathcal{H}_+, \mathcal{H}_-)$  such that

$$I - P_N = \frac{1}{2} \begin{pmatrix} V^*V & V^* \\ V & VV^* \end{pmatrix}.$$

Since  $e^{i\varepsilon}V$  is also a partial isometry (for every  $\varepsilon > 0$ ), there is an orthogonal projection  $Q_\varepsilon$  such that

$$I - Q_\varepsilon = \frac{1}{2} \begin{pmatrix} V^*V & e^{-i\varepsilon}V^* \\ e^{i\varepsilon}V & VV^* \end{pmatrix},$$

so that  $(I - Q_\varepsilon)J(I - Q_\varepsilon) = 0$ . It is clear that  $\|P_N - Q_\varepsilon\| \rightarrow 0$  as  $\varepsilon \rightarrow 0$ .

Since  $\|P_R P_N\| < 1$  and  $\|(I - P_R)(I - P_N)\| < 1$ , there exists  $\varepsilon_0 > 0$  such that

$$\|P_R Q_\varepsilon\| < 1 \quad \text{and} \quad \|(I - P_R)(I - Q_\varepsilon)\| < 1 \quad \text{for } 0 < \varepsilon \leq \varepsilon_0.$$

Hence, there is a projection  $P_\varepsilon \in L(\mathcal{H})$  with  $R(P_\varepsilon) = R(P)$  and  $N(P_\varepsilon) = R(Q_\varepsilon)$ , see Remark 2.1. Then, by Lemma 2.3,  $P_\varepsilon P_\varepsilon^\# = P_\varepsilon^\# P_\varepsilon = 0$ . Finally,  $P_\varepsilon$  can be represented as:

$$P_\varepsilon = P_R(P_R + Q_\varepsilon)^{-1},$$

see (2.1). So,  $P_\varepsilon \neq P$  for every  $0 < \varepsilon \leq \varepsilon_0$ , and  $\|P_\varepsilon - P\| \rightarrow 0$  as  $\varepsilon \rightarrow 0$ .  $\square$

**Corollary 5.7.** *Suppose that a (non-trivial) projection  $P \in L(\mathcal{H})$ , satisfies  $PP^\# = P^\#P = 0$ . Then, there exists a one-parameter family of (different)  $J$ -normal projections  $P_\varepsilon \in L(\mathcal{H})$  onto  $R(P)$  (for  $0 < \varepsilon < \varepsilon_0$ ) such that*

$$c(N(P), N(P_\varepsilon)) \longrightarrow 1 \quad \text{as } \varepsilon \rightarrow 0.$$

*Proof.* Consider the projections  $P_\varepsilon$  obtained in Proposition 5.6. Following the notations in the proof above,  $N(P) = R(P_N)$  and  $N(P_\varepsilon) = R(Q_\varepsilon)$ . Then,

$$c(N(P), N(P_\varepsilon)) = c(R(P_N), R(Q_\varepsilon)) = c(R(I - P_N), R(I - Q_\varepsilon)),$$

because  $P_N$  and  $Q_\varepsilon$  are orthogonal projections. By Remark 2.1,

$$\begin{aligned} & c(R(I - P_N), R(I - Q_\varepsilon))^2 = \\ &= \|(I - Q_\varepsilon)(I - P_N)\|^2 = \|(I - Q_\varepsilon)(I - P_N)(I - Q_\varepsilon)\| = \\ &= \frac{|(1 + e^{i\varepsilon})(1 + e^{-i\varepsilon})|}{4} \left\| \frac{1}{2} \begin{pmatrix} V^*V & \frac{1+e^{-i\varepsilon}}{1+e^{i\varepsilon}}V^* \\ \frac{1+e^{i\varepsilon}}{1+e^{-i\varepsilon}}V & VV^* \end{pmatrix} \right\| = \\ &= \frac{|(1 + e^{i\varepsilon})(1 + e^{-i\varepsilon})|}{4} = \frac{1 + \cos(\varepsilon)}{2} = \cos^2\left(\frac{\varepsilon}{2}\right). \end{aligned}$$

Therefore,  $c(N(P), N(P_\varepsilon)) = \cos(\frac{\varepsilon}{2}) \longrightarrow 1$  as  $\varepsilon \rightarrow 0$ .  $\square$

### $J$ -normal projections with prescribed $J$ -neutral range

Let  $\mathcal{N}$  be a  $J$ -neutral subspace of a Krein space  $\mathcal{H}$  with fundamental symmetry  $J$ . Along these paragraphs, a parametrization for the set of  $J$ -normal projections onto  $\mathcal{N}$  is presented. These results are generalized to an arbitrary pseudo-regular subspace in Section 6.

According to the orthogonal decomposition  $\mathcal{H} = \mathcal{N} \oplus \mathcal{N}^\perp$ , the symmetry  $J$  can be written as a block-operator-matrix

$$J = \begin{pmatrix} 0 & a \\ a^* & b \end{pmatrix} \begin{matrix} \mathcal{N} \\ \mathcal{N}^\perp \end{matrix} \quad (5.1)$$

where  $a \in L(\mathcal{N}^\perp, \mathcal{N})$  and  $b = b^* \in L(\mathcal{N}^\perp)$  satisfy

$$aa^* = I_{\mathcal{N}}, \quad ab = 0 \quad \text{and} \quad a^*a + b^2 = I_{\mathcal{N}^\perp}. \quad (5.2)$$

Since  $a \in L(\mathcal{N}^\perp, \mathcal{N})$  is a coisometry, it follows that  $a^* \in L(\mathcal{N}, \mathcal{N}^\perp)$  is a partial isometry with final space:

$$R(a^*a) = R(a^*) = J(\mathcal{N}).$$

Thus,  $a^*a \in L(\mathcal{N}^\perp)$  is the orthogonal projection onto  $J(\mathcal{N})$ .

On the other hand, if  $P$  is a projection with range  $\mathcal{N}$  then  $P$  can be written as a block-operator-matrix

$$P = \begin{pmatrix} I & x \\ 0 & 0 \end{pmatrix},$$

with  $x \in L(\mathcal{N}^\perp, \mathcal{N})$ . Furthermore,  $P$  satisfies  $PP^\# = 0$  if and only if

$$0 = \begin{pmatrix} I & x \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & a \\ a^* & b \end{pmatrix} \begin{pmatrix} I & 0 \\ x^* & 0 \end{pmatrix} = \begin{pmatrix} ax^* + xa^* + xbx^* & 0 \\ 0 & 0 \end{pmatrix},$$

or equivalently,  $x \in L(\mathcal{N}^\perp, \mathcal{N})$  is a solution of the equation

$$ax^* + xa^* + xbx^* = 0. \quad (5.3)$$

Thus, in order to describe the set of  $J$ -normal projections onto the  $J$ -neutral subspace  $\mathcal{N}$ , the above equation has to be solved. The following result provides a parametrization for the set of solutions of (5.3).

**Lemma 5.8.** *Let  $\mathcal{N}$  be a  $J$ -neutral subspace of  $\mathcal{H}$ . Then,  $x \in L(\mathcal{N}^\perp, \mathcal{N})$  is a solution of (5.3) if and only if there exist operators  $A \in L(\mathcal{N})$  and  $B \in L(\mathcal{N}^\perp, \mathcal{N})$  such that  $A$  is antihermitian,  $J(\mathcal{N}) \subseteq N(B)$  and*

$$x = (A - \frac{1}{2}BbB^*)a + B.$$

*Proof.* Recall that the operators  $a$  and  $b$  considered in (5.3) satisfy the conditions in (5.2). First, suppose that  $x \in L(\mathcal{N}^\perp, \mathcal{N})$  is a solution of (5.3). Since  $a^*a + b^2 = I_{\mathcal{N}^\perp}$ ,  $x$  can be written as  $x = x_1 + x_2$ , where  $x_1 = xa^*a$  and  $x_2 = xb^2$ .

Observe that  $x_2a^* = x_1b = 0$ . Thus,  $0 = ax^* + xa^* + xbx^* = ax_1^* + x_1a^* + x_2bx_2^*$ . In other words,

$$2\operatorname{Re}(x_1a^*) = ax_1^* + x_1a^* = -x_2bx_2^*.$$

So, the antihermitian operator  $A = i\operatorname{Im}(x_1a^*) \in L(\mathcal{N})$  satisfies

$$x_1 = x_1a^*a = (A - \frac{1}{2}x_2bx_2^*)a.$$

Then, considering  $B = x_2 = x(I_{\mathcal{N}^\perp} - a^*a) \in L(\mathcal{N}^\perp, \mathcal{N})$  it follows that  $J(\mathcal{N}) \subseteq N(B)$  and

$$x = (A - \frac{1}{2}BbB^*)a + B.$$

Conversely, given an antihermitian operator  $A \in L(\mathcal{N})$  and  $B \in L(\mathcal{N}^\perp, \mathcal{N})$  such that  $J(\mathcal{N}) \subseteq N(B)$ , consider

$$x := (A - \frac{1}{2}BbB^*)a + B.$$

Then, it is easy to see that  $xa^* = A - \frac{1}{2}BbB^*$  and  $xbx^* = BbB^*$ . Therefore,

$$xa^* + ax^* + xbx^* = (A - \frac{1}{2}BbB^*) + (-A - \frac{1}{2}BbB^*) + BbB^* = 0,$$

i.e.  $x \in L(\mathcal{N}^\perp, \mathcal{N})$  is a solution of (5.3).  $\square$

**Proposition 5.9.** *Let  $\mathcal{N}$  be a  $J$ -neutral subspace of  $\mathcal{H}$ . Then,  $P \in L(\mathcal{H})$  is a  $J$ -normal projection onto  $\mathcal{N}$  if and only if there exist  $A = -A^* \in L(\mathcal{N})$  and  $B \in L(\mathcal{N}^\perp, \mathcal{N})$  with  $J(\mathcal{N}) \subseteq N(B)$  such that*

$$P = \begin{pmatrix} I & (A - \frac{1}{2}BbB^*)a + B \\ 0 & 0 \end{pmatrix},$$

according to the orthogonal decomposition  $\mathcal{H} = \mathcal{N} \oplus \mathcal{N}^\perp$ .

## 6 A parametrization for the set of $J$ -normal projections

Let  $\mathcal{S}$  be a pseudo-regular subspace of a Krein space  $\mathcal{H}$  with fundamental symmetry  $J$ , and denote

$$\mathcal{Q}_{\mathcal{S}} = \{Q \in L(\mathcal{H}) : Q^2 = Q, QQ^\# = Q^\#Q \text{ and } R(Q) = \mathcal{S}\}.$$

The aim of this section is to present an explicit parametrization of  $\mathcal{Q}_{\mathcal{S}}$ . First, notice that there are as many projections in  $\mathcal{Q}_{\mathcal{S}}$  as in  $\mathcal{Q}_{\mathcal{S}^\circ}$ .

**Lemma 6.1.** *Suppose that  $\mathcal{S}$  is a pseudo-regular subspace of  $\mathcal{H}$ . If  $P$  is a  $J$ -normal projection onto  $\mathcal{S}^\circ$  then there is a unique  $J$ -normal projection  $Q$  onto  $\mathcal{S}$  such that  $P$  is the neutral part of  $Q$ , i.e.  $P = Q(I - Q)^\#$ .*

*Proof.* Suppose that  $\mathcal{S}$  is a pseudo-regular subspace of  $\mathcal{H}$  and consider  $\mathcal{T} = \mathcal{S} \cap N(P)$ . Since  $P$  is a projection onto  $\mathcal{S}^\circ \subseteq \mathcal{S}$ , given  $s \in \mathcal{S}$ ,  $(I - P)s \in \mathcal{S} + \mathcal{S}^\circ = \mathcal{S}$ . So that,  $(I - P)s \in \mathcal{S} \cap N(P)$ . Therefore,

$$\mathcal{S} = \mathcal{S}^\circ \dot{+} \mathcal{T}.$$

Then, by Proposition 4.1,  $\mathcal{T}$  is a regular subspace of  $\mathcal{H}$ . Let  $E$  be the  $J$ -selfadjoint projection onto  $\mathcal{T}$ .

Notice that  $EP = 0$  because  $\mathcal{S}^\circ \subseteq \mathcal{S}^{[\perp]} \subseteq \mathcal{T}^{[\perp]}$ . On the other hand,  $R(E) = \mathcal{T} \subseteq N(P)$ . So,  $PE = 0$  and, since  $E$  is  $J$ -selfadjoint, the following commutativity relations have been established:

$$EP = PE = 0 \quad \text{and} \quad EP^\# = P^\#E = 0.$$

Now, define  $Q = E + P$ . Then, by Theorem 3.1,  $Q$  is a  $J$ -normal projection and  $P = Q - E = Q - QQ^\# = Q(I - Q^\#)$ .

Finally, suppose that there is another  $J$ -normal projection  $Q' \in L(\mathcal{H})$  onto  $\mathcal{S}$  such that  $P = Q'(I - Q')^\#$ . Then,  $E' = Q' - P = Q'(Q')^\#$  is a  $J$ -selfadjoint



projection onto a subspace of  $\mathcal{S}$ . Notice that  $R(E') \subseteq N(P)$  because  $PE' = 0$ . Hence,  $R(E') \subseteq \mathcal{T}$ . But,

$$R(E') \dot{+} \mathcal{S}^\circ = \mathcal{S} = \mathcal{T} \dot{+} \mathcal{S}^\circ.$$

Thus,  $R(E') = \mathcal{T}$  and, by the uniqueness of the  $J$ -selfadjoint projection onto a regular subspace,  $E' = E$ .  $\square$

**Theorem 6.2.** *Given a pseudo-regular subspace  $\mathcal{S}$  of  $\mathcal{H}$  with isotropic part  $\mathcal{S}^\circ$ , there is a (continuous) bijection between  $\mathcal{Q}_{\mathcal{S}}$  and  $\mathcal{Q}_{\mathcal{S}^\circ}$ .*

*Proof.* For a fixed pseudo-regular subspace  $\mathcal{S}$  of  $\mathcal{H}$ , let  $\Phi : \mathcal{Q}_{\mathcal{S}} \rightarrow \mathcal{Q}_{\mathcal{S}^\circ}$  be defined by

$$\Phi(Q) = Q(I - Q^\#).$$

It follows by the above lemma that  $\Phi$  is bijective, because for every  $P \in \mathcal{Q}_{\mathcal{S}^\circ}$  there exists a unique  $Q \in \mathcal{Q}_{\mathcal{S}}$  such that  $\Phi(Q) = P$ .  $\square$

**Corollary 6.3.** *Let  $\mathcal{S}$  be a pseudo-regular subspace of a Krein space  $\mathcal{H}$  with fundamental symmetry  $J$ . Then, there is a unique  $J$ -normal projection  $Q$  onto  $\mathcal{S}$  if and only if  $\mathcal{S}^\circ = \{0\}$ . Moreover, in this case  $Q$  is  $J$ -selfadjoint.*

*Proof.* If  $\mathcal{S}^\circ = \{0\}$  then  $\mathcal{S}$  is a regular subspace and there exists a (unique)  $J$ -selfadjoint projection onto  $\mathcal{S}$ . Moreover, if  $Q$  is a  $J$ -normal projection onto  $\mathcal{S}$  then, by Theorem 3.1,  $Q = E + P$  where  $E$  is  $J$ -selfadjoint and  $P$  is a projection onto  $\mathcal{S}^\circ = \{0\}$ . Thus,  $P = 0$  and  $Q = E$ .

On the other hand, if  $\mathcal{S}^\circ \neq \{0\}$  then, as a consequence of Theorem 6.2 and Proposition 5.6, there are infinitely many  $J$ -normal projections onto  $\mathcal{S}$ .  $\square$

By Proposition 4.1, for a fixed pseudo-regular subspace  $\mathcal{S}$  of  $\mathcal{H}$ , if  $\mathcal{S}^\circ$  is the isotropic part of  $\mathcal{S}$  and  $\mathcal{M}$  is a subspace of  $\mathcal{S}$  such that  $\mathcal{S} = \mathcal{S}^\circ \dot{+} \mathcal{M}$  (i.e.  $\mathcal{M}$  is a complement of  $\mathcal{S}^\circ$  in  $\mathcal{S}$ ), then  $\mathcal{M}$  is a regular subspace of  $\mathcal{H}$ . Hence, consider

$$\mathcal{Q}_{\mathcal{S}, \mathcal{M}} = \{Q \in \mathcal{Q}_{\mathcal{S}} : QQ^\# = E_{\mathcal{M}}\},$$

where  $E_{\mathcal{M}}$  stands for the  $J$ -selfadjoint projection onto  $\mathcal{M}$ .

Notice that  $\mathcal{Q}_{\mathcal{S}}$  can be written as the disjoint union of the family  $\mathcal{Q}_{\mathcal{S}, \mathcal{M}}$ , as  $\mathcal{M}$  varies on the complements of  $\mathcal{S}^\circ$  in  $\mathcal{S}$ :

**Lemma 6.4.** *If  $\mathcal{S}$  is a pseudo-regular subspace of  $\mathcal{H}$ , then*

$$\mathcal{Q}_{\mathcal{S}} = \dot{\bigcup}_{\{\mathcal{M} : \mathcal{S} = \mathcal{S}^\circ \dot{+} \mathcal{M}\}} \mathcal{Q}_{\mathcal{S}, \mathcal{M}}, \quad (6.1)$$

where  $\dot{\bigcup}$  denotes a disjoint union.

*Proof.* It is obvious that  $\mathcal{Q}_{\mathcal{S}} = \bigcup_{\{\mathcal{M} : \mathcal{S} = \mathcal{S}^\circ \dot{+} \mathcal{M}\}} \mathcal{Q}_{\mathcal{S}, \mathcal{M}}$ . Suppose that  $Q \in \mathcal{Q}_{\mathcal{S}, \mathcal{M}_1} \cap \mathcal{Q}_{\mathcal{S}, \mathcal{M}_2}$ , where  $\mathcal{M}_1$  and  $\mathcal{M}_2$  are regular subspaces of  $\mathcal{H}$ . Then,

$$E_{\mathcal{M}_1} = QQ^\# = E_{\mathcal{M}_2},$$

or equivalently,  $\mathcal{M}_1 = \mathcal{M}_2$ . Hence,  $\mathcal{Q}_{\mathcal{S}, \mathcal{M}_1} = \mathcal{Q}_{\mathcal{S}, \mathcal{M}_2}$ .  $\square$

## Parametrizing the deck $\mathcal{Q}_{\mathcal{S}, \mathcal{M}}$ for a pseudo-regular subspace $\mathcal{S}$

The following paragraphs are devoted to studying those  $J$ -normal projections onto  $\mathcal{S}$  which have a fixed regular part. Along this section operators are treated as block-operator matrices according to the orthogonal decomposition

$$\mathcal{H} = \mathcal{S}^\circ \oplus (\mathcal{S} \ominus \mathcal{S}^\circ) \oplus \mathcal{S}^\perp,$$

and  $P_{\mathcal{S}^\perp}$ ,  $P_{\mathcal{S}^\circ}$  and  $P_{\mathcal{S} \ominus \mathcal{S}^\circ}$  denote the orthogonal projections onto  $\mathcal{S}^\perp$ ,  $\mathcal{S}^\circ$  and  $\mathcal{S} \ominus \mathcal{S}^\circ$ , respectively.

If  $\mathcal{M}$  is a regular subspace of  $\mathcal{H}$  such that  $\mathcal{S} = \mathcal{S}^\circ[+] \mathcal{M}$ , it is necessary to describe the fundamental symmetry  $J$  and the  $J$ -selfadjoint projection  $E_{\mathcal{M}}$  onto  $\mathcal{M}$  as block-operator matrices.

**Lemma 6.5.** *If  $\mathcal{S}$  is a pseudo-regular subspace of  $\mathcal{H}$ , then  $J$  is represented as the block-operator matrix*

$$J = \begin{pmatrix} 0 & 0 & a \\ 0 & b & c \\ a^* & c^* & d \end{pmatrix} \begin{matrix} \mathcal{S}^\circ \\ \mathcal{S} \ominus \mathcal{S}^\circ \\ \mathcal{S}^\perp \end{matrix}, \quad (6.2)$$

where  $a \in L(\mathcal{S}^\perp, \mathcal{S}^\circ)$ ,  $b = b^* \in GL(\mathcal{S} \ominus \mathcal{S}^\circ)$ ,  $c \in L(\mathcal{S}^\perp, \mathcal{S} \ominus \mathcal{S}^\circ)$  and  $d = d^* \in L(\mathcal{S}^\perp)$  satisfy the following equations:

$$\begin{cases} aa^* &= I_{\mathcal{S}^\circ} \\ b^2 + cc^* &= I_{\mathcal{S} \ominus \mathcal{S}^\circ} \\ a^*a + c^*c + d^2 &= I_{\mathcal{S}^\perp} \\ bc + cd = ad = ac^* &= 0 \end{cases}. \quad (6.3)$$

*Proof.* Notice that  $P_{\mathcal{S}^\circ}JP_{\mathcal{S}^\circ} = 0$  because  $\mathcal{S}^\circ$  is  $J$ -neutral. Also,  $P_{\mathcal{S}^\circ}JP_{\mathcal{S} \ominus \mathcal{S}^\circ} = 0$  because  $\mathcal{S} \ominus \mathcal{S}^\circ \subseteq \mathcal{S}$  and  $\mathcal{S}^\circ \subseteq \mathcal{S}^\perp$ . Then,

$$J = \begin{pmatrix} 0 & 0 & a \\ 0 & b & c \\ a^* & c^* & d \end{pmatrix}.$$

On the other hand, the system of equations (6.3) follows from  $J^2 = I$ .

By Proposition 4.1,  $\mathcal{S} \ominus \mathcal{S}^\circ$  is a regular subspace of  $\mathcal{H}$ . Furthermore, the regularity of  $\mathcal{S} \ominus \mathcal{S}^\circ$  is equivalent to the range inclusion

$$R(c) \subseteq R(b),$$

see [7, Prop. 3.3]. Then, the second equation in (6.3) implies that  $\mathcal{S} \ominus \mathcal{S}^\circ \subseteq R(b)$ . Hence,  $b$  is an invertible selfadjoint operator in  $L(\mathcal{S} \ominus \mathcal{S}^\circ)$ .  $\square$

*Remark 6.6.* Observe that the operator  $a \in L(\mathcal{S}^\perp, \mathcal{S}^\circ)$  appearing in the above lemma is a coisometry. Then,  $a^* \in L(\mathcal{S}^\circ, \mathcal{S}^\perp)$  is a partial isometry with final space  $J(\mathcal{S}^\circ)$ .

Indeed, by the block-operator matrix representation of  $J$  given in (6.2), it is easy to see that  $R(a^*) = J(\mathcal{S}^\circ)$ . Hence,

$$R(a^*a) = R(a^*) = J(\mathcal{S}^\circ). \quad (6.4)$$

Thus,  $a^*a \in L(\mathcal{S}^\perp)$  is the orthogonal projection onto  $J(\mathcal{S}^\circ)$ .

The following lemma presents a block-matrix representation for the  $J$ -selfadjoint projection  $E_{\mathcal{M}}$  onto a particular complement  $\mathcal{M}$  of  $\mathcal{S}^\circ$  in  $\mathcal{S}$ . This is a technical tool necessary to parametrize the deck  $\mathcal{Q}_{\mathcal{S}, \mathcal{M}}$ .

**Lemma 6.7.** *Given a pseudo-regular subspace  $\mathcal{S}$  of  $\mathcal{H}$ , let  $\mathcal{M}$  be a complement of  $\mathcal{S}^\circ$  in  $\mathcal{S}$ . Then, the  $J$ -selfadjoint projection onto  $\mathcal{M}$  is*

$$E_{\mathcal{M}} = \begin{pmatrix} 0 & ar^*b & ar^*(c+br) \\ 0 & I & b^{-1}c+r \\ 0 & 0 & 0 \end{pmatrix}, \quad (6.5)$$

where  $r = P_{\mathcal{S} \ominus \mathcal{S}^\circ} E_{\mathcal{M}} P_{J(\mathcal{S}^\circ)}|_{\mathcal{S}^\perp} \in L(\mathcal{S}^\perp, \mathcal{S} \ominus \mathcal{S}^\circ)$ .

*Proof.* Suppose that  $\mathcal{S}$  is a pseudo-regular subspace of  $\mathcal{H}$ . Then, by Proposition 4.1,  $\mathcal{M}$  is regular.

Denote by  $E_{\mathcal{M}}$  the  $J$ -selfadjoint projection onto  $\mathcal{M}$ . Since  $R(E_{\mathcal{M}}) = \mathcal{M} \subseteq \mathcal{S}$  it follows that  $P_{\mathcal{S}^\perp} E_{\mathcal{M}} = 0$ , so that the third row in the matrix representation of  $E_{\mathcal{M}}$  is zero. Also, since  $\mathcal{S}^\circ \subseteq \mathcal{S}^{[\perp]} \subseteq \mathcal{M}^{[\perp]} = N(E_{\mathcal{M}})$ , it follows that  $E_{\mathcal{M}} P_{\mathcal{S}^\circ} = 0$ . So that the first column is also zero. Therefore,

$$E_{\mathcal{M}} = \begin{pmatrix} 0 & u & v \\ 0 & p & q \\ 0 & 0 & 0 \end{pmatrix},$$

where  $u \in L(\mathcal{S} \ominus \mathcal{S}^\circ, \mathcal{S}^\circ)$ ,  $v \in L(\mathcal{S}^\perp, \mathcal{S}^\circ)$ ,  $p \in L(\mathcal{S} \ominus \mathcal{S}^\circ)$  and  $q \in L(\mathcal{S}^\perp, \mathcal{S} \ominus \mathcal{S}^\circ)$  satisfy

$$\begin{cases} up &= u \\ uq &= v \\ p^2 &= p \\ pq &= q \end{cases}.$$

Thus,  $p = P_{\mathcal{S} \ominus \mathcal{S}^\circ} E_{\mathcal{M}}|_{\mathcal{S} \ominus \mathcal{S}^\circ}$  is a projection with

$$R(p) = P_{\mathcal{S} \ominus \mathcal{S}^\circ} E_{\mathcal{M}}(\mathcal{S} \ominus \mathcal{S}^\circ) = P_{\mathcal{S} \ominus \mathcal{S}^\circ} E_{\mathcal{M}}(\mathcal{S}) = P_{\mathcal{S} \ominus \mathcal{S}^\circ}(\mathcal{M}) = P_{\mathcal{S} \ominus \mathcal{S}^\circ}(\mathcal{S}) = \mathcal{S} \ominus \mathcal{S}^\circ,$$

because  $\mathcal{S}^\circ \subseteq N(P_{\mathcal{S} \ominus \mathcal{S}^\circ}) \cap N(E_{\mathcal{M}})$ . Hence,  $p = I_{\mathcal{S} \ominus \mathcal{S}^\circ}$ .

Furthermore,  $E_{\mathcal{M}}$  is  $J$ -selfadjoint if and only if

$$JE_{\mathcal{M}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & b & bq \\ 0 & a^*u + c^* & (a^*u + c^*)q \end{pmatrix}$$

is selfadjoint, or equivalently, if

$$a^*u + c^* = q^*b. \quad (6.6)$$

By (6.3),  $aa^* = I_{\mathcal{S}^\circ}$  and  $ac^* = 0$ . Thus, multiplying on the left by  $a$ , it follows that  $u = aq^*b$ . Thus,

$$E_{\mathcal{M}} = \begin{pmatrix} 0 & aq^*b & aq^*bq \\ 0 & I & q \\ 0 & 0 & 0 \end{pmatrix},$$

where  $q = P_{\mathcal{S} \ominus \mathcal{S}^\circ} E_{\mathcal{M}}|_{\mathcal{S}^\perp}$ . Replacing  $u$  in (6.6), notice that  $q$  satisfies  $a^*aq^*b + c^* = q^*b$ , or equivalently,

$$q = q(a^*a) + b^{-1}c.$$

Therefore, if  $r = q(a^*a)$  then  $aq^*b = a(c^*b^{-1} + r^*)b = ar^*b$ , and (6.5) follows.  $\square$

Finally, a block-matrix representation of a projection  $Q \in L(\mathcal{H})$  onto  $\mathcal{S}$  is needed. Since  $R(Q) = \mathcal{S}$ , observe that  $P_{\mathcal{S}^\circ}QP_{\mathcal{S}^\circ} = P_{\mathcal{S}^\circ}$ ,  $P_{\mathcal{S} \ominus \mathcal{S}^\circ}QP_{\mathcal{S} \ominus \mathcal{S}^\circ} = P_{\mathcal{S} \ominus \mathcal{S}^\circ}$  and

$$P_{\mathcal{S}^\circ}QP_{\mathcal{S} \ominus \mathcal{S}^\circ} = P_{\mathcal{S} \ominus \mathcal{S}^\circ}QP_{\mathcal{S}^\circ} = 0.$$

Then,  $Q$  is represented as the block-operator matrix

$$Q = \begin{pmatrix} I & 0 & x \\ 0 & I & y \\ 0 & 0 & 0 \end{pmatrix}, \quad (6.7)$$

where  $x = P_{\mathcal{S}^\circ}Q|_{\mathcal{S}^\perp} \in L(\mathcal{S}^\perp, \mathcal{S}^\circ)$  and  $y = P_{\mathcal{S} \ominus \mathcal{S}^\circ}Q|_{\mathcal{S}^\perp} \in L(\mathcal{S}^\perp, \mathcal{S} \ominus \mathcal{S}^\circ)$ .

Furthermore, if  $Q \in \mathcal{Q}_{\mathcal{S}, \mathcal{M}}$  then, by Theorem 3.1,  $P = Q - E_{\mathcal{M}}$  is a projection onto  $\mathcal{S}^\circ$  such that  $PP^\# = P^\#P = 0$ . Moreover, by (6.5),  $P$  has the form

$$P = Q - E_{\mathcal{M}} = \begin{pmatrix} I & -ar^*b & x - ar^*(c + br) \\ 0 & 0 & y - b^{-1}c - r \\ 0 & 0 & 0 \end{pmatrix}.$$

But,  $R(P) = \mathcal{S}^\circ$  if and only if

$$y = b^{-1}c + r.$$

Also,  $PP^\# = 0$  if and only if  $PJP^* = 0$ , or equivalently,

$$\begin{pmatrix} I & -ar^*b & z \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & a \\ 0 & b & c \\ a^* & c^* & d \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ -bra^* & 0 & 0 \\ z^* & 0 & 0 \end{pmatrix} = 0,$$

where  $z = x - ar^*(c + br)$ . But the above equation is equivalent to

$$z(I - r^*bc)^*a^* + a(I - r^*bc)z^* + zdz^* + ar^*b^3ra^* = 0. \quad (6.8)$$

The following lemma is devoted to describe the solutions of (6.8), where  $a$ ,  $b$ ,  $c$ ,  $d$  and  $r$  are the operators appearing in (6.2) and in (6.5).

**Lemma 6.8.** *An operator  $z \in L(\mathcal{S}^\perp, \mathcal{S}^\circ)$  is a solution of (6.8) if and only if there exist  $A = -A^* \in L(\mathcal{S}^\circ)$  and  $B \in L(\mathcal{S}^\perp, \mathcal{S}^\circ)$  with  $J(\mathcal{S}^\circ) \subseteq N(B)$  such that*

$$z = (A + \operatorname{Re}(Bc^*bra^*) - \frac{1}{2}(BdB^* + ar^*b^3ra^*))a + B.$$

*Proof.* Let  $z \in L(\mathcal{S}^\perp, \mathcal{S}^\circ)$  be a solution of (6.8) and consider the operators

$$z_1 = z(a^*a) \quad \text{and} \quad z_2 = z(I_{\mathcal{S}^\perp} - a^*a).$$

Notice that  $z_1(I - r^*bc)^*a^* + a(I - r^*bc)z_1^* = z_1a^* + az_1^* = 2\operatorname{Re}(z_1a^*)$  because  $ac^* = ca^* = 0$ . Also,

$$z_2(I - r^*bc)^*a^* + a(I - r^*bc)z_2^* = -z_2c^*bra^* - ar^*bcz_2^* = -2\operatorname{Re}(z_2c^*bra^*),$$

because  $z_2a^* = az_2^* = 0$ . On the other hand, since  $ad = da^* = 0$  it is easy to see that

$$zdz^* = (z_1 + z_2)d(z_1 + z_2)^* = z_2dz_2^*.$$

Therefore, (6.8) is equivalent to

$$2\operatorname{Re}(z_1a^*) = 2\operatorname{Re}(z_2c^*bra^*) - z_2dz_2^* - ar^*b^3ra^*. \quad (6.9)$$

Then, considering the antihermitian operator  $A = i\operatorname{Im}(z_1a^*) \in L(\mathcal{S}^\circ)$ , it follows that

$$\begin{aligned} z_1 &= (z_1a^*)a = (i\operatorname{Im}(z_1a^*) + \operatorname{Re}(z_1a^*))a \\ &= (A + \operatorname{Re}(z_2c^*bra^*) - \frac{1}{2}(z_2dz_2^* + ar^*b^3ra^*))a. \end{aligned}$$

Hence,  $B = z_2 \in L(\mathcal{S}^\perp, \mathcal{S}^\circ)$  satisfies  $J(\mathcal{S}^\circ) \subseteq N(B)$  and

$$z = z_1 + z_2 = (A + \operatorname{Re}(Bc^*bra^*) - \frac{1}{2}(BdB^* + ar^*b^3ra^*))a + B.$$

Conversely, given an antihermitian operator  $A \in L(\mathcal{S}^\circ)$  and  $B \in L(\mathcal{S}^\perp, \mathcal{S}^\circ)$  such that  $N(b)^\perp \subseteq N(d)$ , consider

$$z_{A,B} := (A + \operatorname{Re}(Bc^*bra^*) - \frac{1}{2}(BdB^* + ar^*b^3ra^*))a + B.$$

Then, it is easy to see that  $z_{A,B} \in L(\mathcal{S}^\perp, \mathcal{S}^\circ)$  is a solution of (6.8).  $\square$

Finally, it is possible to parametrize the deck  $\mathcal{Q}_{\mathcal{S}, \mathcal{M}}$  as follows:

**Theorem 6.9.** *Let  $Q \in L(\mathcal{H})$  be a projection onto a pseudo-regular subspace  $\mathcal{S}$  of  $\mathcal{H}$ . Suppose that  $\mathcal{M}$  is a regular subspace of  $\mathcal{H}$  such that  $\mathcal{S} = \mathcal{S}^\circ \dot{+} \mathcal{M}$ . Then,  $Q \in \mathcal{Q}_{\mathcal{S}, \mathcal{M}}$  if and only if*

$$Q = \begin{pmatrix} I & 0 & (A + \operatorname{Re}(Bc^*bra^*) - \frac{1}{2}(BdB^* + ar^*b^3ra^*))a + B + ar^*(c + br) \\ 0 & I & b^{-1}c + r \\ 0 & 0 & 0 \end{pmatrix}, \quad (6.10)$$

where  $r = P_{\mathcal{S} \ominus \mathcal{S}^\circ} E_{\mathcal{M}}(a^*a) \in L(\mathcal{S}^\perp, \mathcal{S} \ominus \mathcal{S}^\circ)$ ,  $A = -A^* \in L(\mathcal{S}^\circ)$  and  $B \in L(\mathcal{S}^\perp, \mathcal{S}^\circ)$  is such that  $J(\mathcal{S}^\circ) \subseteq N(B)$ .

*Proof.* Suppose that  $Q \in \mathcal{Q}_{\mathcal{S}, \mathcal{M}}$ , i.e.  $Q \in L(\mathcal{H})$  is a  $J$ -normal projection onto  $\mathcal{S}$  satisfying  $QQ^\# = Q^\#Q = E_{\mathcal{M}}$ . Then,  $P = Q - E_{\mathcal{M}}$  is a projection onto  $\mathcal{S}^\circ$  such that  $PP^\# = P^\#P = 0$ . Hence, if  $Q$  is written as in (6.7) it follows that  $y = b^{-1}c$ .

Then, by the discussion above,

$$P = \begin{pmatrix} I & -ar^*b & x - ar^*(c + br) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where  $z = x - ar^*(c + br)$  is a solution of (6.8). Thus, by Proposition 6.8, there exist  $A = -A^* \in L(\mathcal{S}^\circ)$  and  $B \in L(\mathcal{S}^\perp, \mathcal{S}^\circ)$  with  $J(\mathcal{S}^\circ) \subseteq N(B)$  such that

$$P = \begin{pmatrix} I & -ar^*b & (A + \operatorname{Re}(Bc^*bra^*) - \frac{1}{2}(BdB^* + ar^*b^3ra^*))a + B \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore,

$$Q = \begin{pmatrix} I & 0 & (A + \operatorname{Re}(Bc^*bra^*) - \frac{1}{2}(BdB^* + ar^*b^3ra^*))a + B + ar^*(c + br) \\ 0 & I & b^{-1}c + r \\ 0 & 0 & 0 \end{pmatrix}.$$

The converse follows immediately.  $\square$

Given a pseudo regular subspace  $\mathcal{S}$  of  $\mathcal{H}$ , denote by  $\mathcal{C}(\mathcal{S}^\circ)$  the set of complements of  $\mathcal{S}^\circ$  in  $\mathcal{S}$ . Recall that, by Lemma 6.4, the set of  $J$ -normal projections onto  $\mathcal{S}$  is decomposed as

$$\mathcal{Q}_{\mathcal{S}} = \bigcup_{\mathcal{M} \in \mathcal{C}(\mathcal{S}^\circ)} \mathcal{Q}_{\mathcal{S}, \mathcal{M}}.$$

Furthermore, for a fixed  $\mathcal{M} \in \mathcal{C}(\mathcal{S}^\circ)$ , Theorem 6.9 states that the deck  $\mathcal{Q}_{\mathcal{S}, \mathcal{M}}$  is parametrized by the bijection  $\Psi_{\mathcal{M}} : \mathcal{AH}(\mathcal{S}^\circ) \times \mathcal{N}_\circ \rightarrow \mathcal{Q}_{\mathcal{S}, \mathcal{M}}$  given by

$$\Psi_{\mathcal{M}}(A, B) = \begin{pmatrix} I & 0 & (A + \operatorname{Re}(Bc^*bra^*) - \frac{1}{2}(BdB^* + ar^*b^3ra^*))a + B + ar^*(c + br) \\ 0 & I & b^{-1}c + r \\ 0 & 0 & 0 \end{pmatrix},$$

where  $\mathcal{AH}(\mathcal{S}^\circ)$  stands for the real vector space of antihermitian operators acting on  $\mathcal{S}^\circ$  and  $\mathcal{N}_\circ$  is the set composed by those operators  $B \in L(\mathcal{S}^\perp, \mathcal{S}^\circ)$  such that  $J(\mathcal{S}^\circ) \subseteq N(B)$ .

Therefore, the set  $\mathcal{Q}_{\mathcal{S}}$  of  $J$ -normal projections onto  $\mathcal{S}$  is parametrized as follows:

**Theorem 6.10.** *Let  $\mathcal{S}$  be a pseudo-regular subspace of  $\mathcal{H}$ . Then, the function  $\Psi : \mathcal{RC}(\mathcal{S}^\circ) \times \mathcal{AH}(\mathcal{S}^\circ) \times \mathcal{N}_\circ \rightarrow \mathcal{Q}_{\mathcal{S}}$  defined by*

$$\Psi(\mathcal{M}, A, B) = \begin{pmatrix} I & 0 & (A + \operatorname{Re}(Bc^*bra^*) - \frac{1}{2}(BdB^* + ar^*b^3ra^*))a + B + ar^*(c + br) \\ 0 & I & b^{-1}c + r \\ 0 & 0 & 0 \end{pmatrix},$$

*is one-to one.*

Observe that in the expression defining  $\Psi$  appears the operator

$$r = P_{\mathcal{S} \ominus \mathcal{S}^\circ} E_{\mathcal{M}} P_{J(\mathcal{S}^\circ)}|_{\mathcal{S}^\perp} \in L(\mathcal{S}^\perp, \mathcal{S} \ominus \mathcal{S}^\circ),$$

given in Lemma 6.7, where  $P_{\mathcal{S} \ominus \mathcal{S}^\circ}$  and  $P_{J(\mathcal{S}^\circ)}$  are the orthogonal projections onto  $\mathcal{S} \ominus \mathcal{S}^\circ$  and  $J(\mathcal{S}^\circ)$ , respectively, and  $E_{\mathcal{M}}$  is the  $J$ -selfadjoint projection onto  $\mathcal{M}$ .

### An interesting particular deck: $\mathcal{Q}_{\mathcal{S}, \mathcal{S} \ominus \mathcal{S}^\circ}$

Let  $\mathcal{S}$  be a fixed pseudo-regular subspace of a Krein space  $\mathcal{H}$  with fundametal symmetry  $J$ . These paragraphs are devoted to describe the set  $\mathcal{Q}_{\mathcal{S}, \mathcal{S} \ominus \mathcal{S}^\circ}$ , i.e. the family of  $J$ -normal projections  $Q \in L(\mathcal{H})$  onto  $\mathcal{S}$  such that  $QQ^\#$  is the  $J$ -selfadjoint projection onto the (regular) subspace  $\mathcal{S} \ominus \mathcal{S}^\circ$ . In this particular deck there is a minimal norm projection, see Remark 6.12.

First of all, since  $\mathcal{S} \ominus \mathcal{S}^\circ$  is a complement of  $\mathcal{S}^\circ$  in  $\mathcal{S}$ , it follows by Lemma 6.7 that the  $J$ -selfadjoint projection onto  $\mathcal{S} \ominus \mathcal{S}^\circ$  (hereafter denoted by  $E$ ) is the block-operator matrix given by (6.5), where

$$r = P_{\mathcal{S} \ominus \mathcal{S}^\circ} E P_{J(\mathcal{S}^\circ)}|_{\mathcal{S}^\perp} \in L(\mathcal{S}^\perp, \mathcal{S} \ominus \mathcal{S}^\circ).$$

But,  $J(\mathcal{S}^\circ) \subseteq J(\mathcal{S}^\circ) + \mathcal{S}^{[\perp]} = J(\mathcal{S}^\circ + \mathcal{S}^\perp) = J((\mathcal{S} \ominus \mathcal{S}^\circ)^\perp) = N(E)$ . Therefore,  $r = 0$  and the block-operator matrix representation of  $E$  is

$$E = \begin{pmatrix} 0 & 0 & 0 \\ 0 & I & b^{-1}c \\ 0 & 0 & 0 \end{pmatrix}.$$

Furthermore, as a consequence of Theorem 6.9,  $\mathcal{Q}_{\mathcal{S}, \mathcal{S} \ominus \mathcal{S}^\circ}$  is parametrized as:

**Proposition 6.11.** *Let  $\mathcal{S}$  be a pseudo-regular subspace of a Krein space  $\mathcal{H}$  with fundametal symmetry  $J$ . A projection  $Q$  onto  $\mathcal{S}$  satisfies  $QQ^\# = Q^\#Q = E$  if and only if*

$$Q = \begin{pmatrix} I & 0 & (A - \frac{1}{2}BdB^*)a + B \\ 0 & I & b^{-1}c \\ 0 & 0 & 0 \end{pmatrix}, \quad (6.11)$$

where  $a, b, c$  and  $d$  are the operators appearing in (6.2),  $A = -A^* \in L(\mathcal{S}^\circ)$  and  $B \in L(\mathcal{S}^\perp, \mathcal{S}^\circ)$  is such that  $J(\mathcal{S}^\circ) \subseteq N(B)$ .

*Remark 6.12.* In this particular case it is possible to estimate

$$\min\{\|Q\| : Q \in \mathcal{Q}_{\mathcal{S}, \mathcal{S} \ominus \mathcal{S}^\circ}\}.$$

Indeed, if  $P_0$  is the orthogonal projection onto  $\mathcal{S}^\circ$  and  $E$  stands for the  $J$ -selfadjoint projection onto  $\mathcal{S} \ominus \mathcal{S}^\circ$ , then  $Q_0 = E + P_0 \in \mathcal{Q}_{\mathcal{S}, \mathcal{S} \ominus \mathcal{S}^\circ}$ . Furthermore,

$$\|Q_0\|^2 = \|Q_0 Q_0^*\| = \|EE^* + P_0\| = \max\{\|EE^*\|, \|P_0\|\} = \|EE^*\| = \|E\|^2,$$

because  $R(EE^*) = \mathcal{S} \ominus \mathcal{S}^\circ$  is orthogonal to  $R(P_0) = \mathcal{S}^\circ$ . Therefore,  $\|Q_0\| = \|E\|$ .

On the other hand, if  $Q \in \mathcal{Q}_{\mathcal{S}, \mathcal{S} \ominus \mathcal{S}^\circ}$  then there exists a (unique)  $P = P^2 \in L(\mathcal{H})$  such that  $PP^\# = P^\#P = 0$  and  $Q = E + P$ .

Consider a sequence  $\{x_n\}_{n \geq 1}$  in the unit ball of  $\mathcal{H}$  such that  $\|Ex_n\| \rightarrow \|E\|$  as  $n \rightarrow \infty$ . Then,

$$\|Q\|^2 \geq \|Qx_n\|^2 = \|Ex_n\|^2 + \|Px_n\|^2 \geq \|Ex_n\|^2 \rightarrow \|E\|^2 = \|Q_0\|^2.$$

Hence,  $\|Q_0\| = \min\{\|Q\| : Q \in \mathcal{Q}_{\mathcal{S}, \mathcal{S} \ominus \mathcal{S}^\circ}\}$ .

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